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Deformation of quantum mechanics in fractional-dimensional space

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Abstract

A new kind of deformed calculus (the D-deformed calculus) that takes place in fractional-dimensional spaces is presented. The D-deformed calculus is shown to be an appropriate tool for treating fractional-dimensional systems in a simple way and quite analogous to their corresponding one-dimensional partners. Two simple systems, the free particle and the harmonic oscillator in fractional-dimensional spaces, are reconsidered in the framework of the D-deformed quantum mechanics. Confined states in a D-deformed quantum well are studied. D-deformed coherent states are also found.

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1. Introduction

Fractional-dimensional space approaches have been shown to be useful in the study of several physical systems. Theoretical schemes dealing with non-integer space dimensionalities have frequently been considered in the study of critical phenomena (see for instance [1, 2]) and of fractal structures [3] or in modelling semiconductor heterostructure systems [4–8].

In the above-mentioned schemes, the fractional dimensionality does not refer to the real space, but to an auxiliary effective environment used to describe the real system. Nevertheless, the idea of a real spacetime having a dimension slightly different from four has also been considered by several authors [9–12]. Actually, the deviations of the spacetime dimension from four have been found to be very small [9–12]. However, the question whether the dimension of the spacetime is an integer or a fractional number constitutes a basic problem not only for its conceptual significance, but also because the possibility that the spacetime dimension is different from four may lead to interesting consequences (e.g. it is well known that a deviation of the spacetime dimension from the value four eliminates the logarithmic

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divergences of quantum electrodynamics, independently of how small the deviation from four may be [13]).

Recently, the existence of some similarities between the so-called N-body Calogero models and the problem corresponding to a fractional-dimensional harmonic oscillator of a single degree of freedom [14] has been shown. This result, together with the remarkable fact that fractional-dimensional bosons can be considered as generalized parabosons [14], suggests new potential applications of the non-integer-dimensional space approaches.

It was shown in [14] that the fractional-dimensional Bose operators together with the reflection operator form an R-deformed Heisenberg algebra with a deformation parameter depending on the dimension of the space. Deformations of the Heisenberg algebra leading to the so-called q-deformed quantum mechanics have been extensively investigated (see for instance [15–18]). Taking into account the results obtained in [14], we develop in the present paper a new deformed calculus (the D-deformed calculus) in analogy to the q-deformed calculus commonly treated in the literature [18–21]. The new calculus allows us to express and solve problems concerning fractional-dimensional systems in a simple way and quite analogous to the corresponding undeformed ($D = 1$) problems. The paper is organized as follows. In section 2 we introduce the D-deformed calculus. The problems corresponding to the free particle and the harmonic oscillator in fractional-dimensional space were studied in [14]. We reconsider these problems in sections 3 and 4 respectively, but now from the point of view of the D-deformed quantum mechanics. Of course, the final results in these sections coincide with the results obtained in [14]. However, in terms of the new D-deformed calculus, the above-mentioned problems can be solved immediately and in an elegant way. In section 5 the single-particle confined states in a fractional-dimensional quantum well are studied. The dimensional dependence of the eigenenergies corresponding to the ground and to the first excited states is shown. In both cases an increase of the energies as the dimension increases is observed. The probability density function describing the motion of the particle confined in the D-deformed quantum well is also studied for varying the dimensionality. The D-deformed coherent states are found in section 6 and conclusions are summarized in section 7.

2. *D*-deformed calculus

It is well known that the one-dimensional momentum operator is given by

$$P = \frac{1}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}\xi} \tag{1}$$

where we have taken $\hbar = 1$. However, in a fractional-dimensional space, because of the inclusion of the integration weight [22]

$$\frac{\sigma(D)}{2}|\xi|^{D-1} \qquad \sigma(D) = \frac{2\pi^{D/2}}{\Gamma(D/2)} \tag{2}$$

this operator is no longer Hermitian. Therefore, a more general momentum operator has to be defined for systems in fractional-dimensional spaces. Starting with the Wigner commutation relations for the canonical variables of a Bose-like oscillator of a single degree of freedom, the fractional-dimensional momentum operator has been found to be [14],

$$P = \frac{1}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}\xi} + \mathrm{i}\frac{(D-1)}{2\xi}R - \mathrm{i}\frac{(D-1)}{2\xi} \tag{3}$$

where R is the reflection operator.

The momentum operator (equation (3)) suggests a deformation of quantum mechanics in fractional-dimensional spaces. Indeed, we can introduce a new D-deformed derivative operator

$$\frac{\mathrm{d}_D}{\mathrm{d}_D\xi} = \frac{\mathrm{d}}{\mathrm{d}\xi} + \frac{(D-1)}{2\xi}(1-R) \tag{4}$$

and then the fractional-dimensional momentum operator (equation (3)) can be rewritten in the standard form

$$P = \frac{1}{\mathrm{i}}\frac{\mathrm{d}_D}{\mathrm{d}_D\xi}. \tag{5}$$

Thus the D-deformed annihilation and creation operators can be defined in the following way,

$$a_D = \frac{1}{\sqrt{2}}\left(\xi + \frac{\mathrm{d}_D}{\mathrm{d}_D\xi}\right) \qquad a_D^{\dagger} = \frac{1}{\sqrt{2}}\left(\xi - \frac{\mathrm{d}_D}{\mathrm{d}_D\xi}\right). \tag{6}$$

The action of these operators is given as [14]

$$a_D|0\rangle = 0 \tag{7}$$

$$a_D|2n\rangle = \sqrt{2n}|2n-1\rangle \qquad a_D|2n+1\rangle = \sqrt{2n+D}|2n\rangle \tag{8}$$

$$a_D^{\dagger}|2n\rangle = \sqrt{2n+D}|2n+1\rangle \qquad a_D^{\dagger}|2n+1\rangle = \sqrt{2n+2}|2n+2\rangle \tag{9}$$

where $n = 0, 1, 2, 3, \ldots$.

Now by introducing the corresponding D-factor (analogue to the q-factor) as

$$[n]_D = n + \frac{(D-1)}{2}(1-(-1)^n) \tag{10}$$

the equations (8) and (9) can be rewritten in the usual form

$$a_D|n\rangle = \sqrt{[n]_D}|n-1\rangle \tag{11}$$

and

$$a_D^{\dagger}|n\rangle = \sqrt{[n+1]_D}|n+1\rangle \tag{12}$$

respectively.

Taking into account equation (10) and in analogy to the q-deformed standard procedures we can define a D-deformed factorial function as

$$[n]_D! = [n]_D[n-1]_D\cdots[1]_D[0]_D! = \begin{cases} \dfrac{2^n\left(\frac{n}{2}\right)!\,\Gamma\left(\frac{n+D}{2}\right)}{\Gamma(D/2)} & \text{for } n \text{ even} \\ \dfrac{2^n\left(\frac{n-1}{2}\right)!\,\Gamma\left(\frac{n+D+1}{2}\right)}{\Gamma(D/2)} & \text{for } n \text{ odd} \end{cases}. \tag{13}$$

This D-deformed factorial function is a particular case of the generalized factorial function [23].

The eigenstates $|n\rangle$ of the operator

$$N_D|n\rangle = n|n\rangle \qquad N_D = \tfrac{1}{2}\{a_D^{\dagger}, a_D\} - D/2 \tag{14}$$

may be obtained by repeated applications of $a_D^{\dagger}$ on the vacuum state $|0\rangle$

$$|n\rangle = \frac{(a_D^{\dagger})^n}{\sqrt{[n]_D!}}|0\rangle. \tag{15}$$

It is easy to prove that in this Fock space, the relations

$$a_D^{\dagger}a_D = [n]_D \qquad a_D a_D^{\dagger} = [n+1]_D \tag{16}$$

take place.

From the definition of the D-deformed derivative (equation (4)) we can introduce a D-deformed integration, so that if

$$\frac{\mathrm{d}_D f(\xi)}{\mathrm{d}_D \xi} = F(\xi) \tag{17}$$

then

$$f(\xi) = \int F(\xi)\,\mathrm{d}_D\xi + \text{const.} \tag{18}$$

With the aim of finding the appropriate expression for the D-deformed integration, we observe that

$$\frac{\mathrm{d}_D f(\xi)}{\mathrm{d}_D \xi} = \left[1 + \frac{(D-1)}{2\xi}(1-R)\int \mathrm{d}\xi\right]\frac{\mathrm{d}f(\xi)}{\mathrm{d}\xi} = F(\xi) \tag{19}$$

and hence

$$\frac{\mathrm{d}f(\xi)}{\mathrm{d}\xi} = \left[1 + \frac{(D-1)}{2\xi}(1-R)\int \mathrm{d}\xi\right]^{-1} F(\xi). \tag{20}$$

From the equation given above it follows

$$f(\xi) = \int F(\xi)\,\mathrm{d}_D\xi = \sum_{n=0}^{\infty}\left[-\int \mathrm{d}\xi\,\frac{(D-1)}{2\xi}(1-R)\right]^n \int \mathrm{d}\xi\, F(\xi). \tag{21}$$

This expression may be rewritten as

$$\int F(\xi)\,\mathrm{d}_D\xi = \sum_{n=0}^{\infty}(-1)^n I_n \tag{22}$$

where the terms I_n satisfy the following recurrence formula

$$I_{n+1} = \int \frac{(D-1)}{2\xi}(1-R) I_n\,\mathrm{d}\xi \qquad I_0 = \int F(\xi)\,\mathrm{d}\xi. \tag{23}$$

With respect to the D-deformed calculus induced by the fractional-dimensional integration weight (equation (2)), the following identities can be easily demonstrated

$$\frac{\mathrm{d}_D[f(\xi)g(\xi)]}{\mathrm{d}_D\xi} = g(\xi)\frac{\mathrm{d}_D f(\xi)}{\mathrm{d}_D\xi} + \frac{\mathrm{d}_D g(\xi)}{\mathrm{d}_D\xi} R f(\xi) + \frac{\mathrm{d}g(\xi)}{\mathrm{d}\xi}(1-R)f(\xi) \tag{24}$$

and after integrating the above equation

$$\int g(\xi)\frac{\mathrm{d}_D f(\xi)}{\mathrm{d}_D\xi}\mathrm{d}_D\xi = f(\xi)g(\xi) - \int \frac{\mathrm{d}_D g(\xi)}{\mathrm{d}_D\xi} R f(\xi)\,\mathrm{d}_D\xi - \int \frac{\mathrm{d}g(\xi)}{\mathrm{d}\xi}(1-R)f(\xi)\,\mathrm{d}_D\xi. \tag{25}$$

One should note that if either $f(\xi)$ or $g(\xi)$ is an even function of ξ, equations (24) and (25) reduce to a D-deformed Leibnitz rule and a D-deformed formula of integration by parts, respectively. This is a consequence of the fact that the D-deformed derivative acts on even functions as the ordinary derivative.

3. *D*-deformed free particle

The eigenstates of the momentum operator corresponding to a free particle of a single degree of freedom in a fractional-dimensional space can be found now in terms of the D-deformed calculus introduced in the previous section. Thus, the eigenstates of the fractional-dimensional momentum operator are determined by the following equation

$$P\Psi_p = -\mathrm{i}\frac{\mathrm{d}_D\Psi_p}{\mathrm{d}_D\xi} = p\Psi_p. \tag{26}$$

The corresponding eigenfunctions are immediately found to be

$$\Psi_p = A_p E_D(\mathrm{i}p\xi) \tag{27}$$

where $E_D(x)$ represents the D-deformed exponential function (see appendix A). The normalization factor A_p can be found from the orthonormalization condition

$$\langle\Psi_p|\Psi_{p'}\rangle = \frac{\sigma(D)}{2}\lim_{\gamma\to 0}\int_{-\infty}^{\infty} \mathrm{e}^{-\gamma\xi^2}\Psi_p^*(\xi)\Psi_{p'}(\xi)|\xi|^{D-1}\,\mathrm{d}\xi = \delta(p-p') \quad (\gamma > 0) \tag{28}$$

in a similar way as in [14]. After the corresponding calculations we arrive at the following expression

$$A_p = \frac{1}{2^{D/2-1}\Gamma(D/2)}\sqrt{\frac{p^{D-1}}{2\sigma(D)}}. \tag{29}$$

One should note that the eigenfunctions Ψ_p describing the motion of a free particle of a single degree of freedom in a fractional-dimensional space can be considered as D-deformed plane waves and they reduce to the ordinary plane de Broglie waves when $D = 1$.

4. *D*-deformed harmonic oscillator

The eigenfunctions in the coordinate representation corresponding to the D-deformed harmonic oscillator can be derived from equation (15) without much difficulty. First we consider the vacuum state $|0\rangle$ which satisfies equation (7). Then, using the expression of a_D in the coordinate representation (equation (6)) we have the following D-deformed differential equation

$$\left(\frac{\mathrm{d}_D}{\mathrm{d}_D\xi} + \xi\right)\chi_0 = 0 \tag{30}$$

where $\chi_0 = \langle\xi \mid 0\rangle$ represents the eigenfunction of the ground state of the D-deformed harmonic oscillator. Now by solving equation (30) we found that

$$\chi_0 = C_0 \exp[-\xi^2/2] \tag{31}$$

where C_0 is a normalization factor. From the normalization condition

$$\frac{\sigma(D)}{2}\int_{-\infty}^{\infty} |\chi_0|^2|\xi|^{D-1}\,\mathrm{d}\xi = 1 \tag{32}$$

the normalization constant is found to be

$$C_0 = \frac{1}{\pi^{D/4}}. \tag{33}$$

Once the ground state is found, the excited states may be calculated from equation (15). Thus the excited states are determined by

$$\chi_n = \langle\xi \mid n\rangle = \frac{C_0}{\sqrt{[n]_D!}}\left[\frac{1}{\sqrt{2}}\left(\xi - \frac{\mathrm{d}_D}{\mathrm{d}_D\xi}\right)\right]^n \exp[-\xi^2/2]. \tag{34}$$

Now if we take into account that

$$(-1)^n \exp[\xi^2/2]\left(\frac{\mathrm{d}_D}{\mathrm{d}_D\xi}\right)^n \exp[-\xi^2] = \left(\xi - \frac{\mathrm{d}_D}{\mathrm{d}_D\xi}\right)^n \exp[-\xi^2/2] \tag{35}$$

a relation that can be demonstrated by induction, the excited states in coordinate representation can be written as

$$\chi_n = \langle \xi \mid n \rangle = \frac{[n]_D! \exp[-\xi^2/2]}{n!\sqrt{\pi^{D/2} 2^n [n]_D!}} H_n^D(\xi) \tag{36}$$

where

$$H_n^D(\xi) = \frac{n!}{[n]_D!}(-1)^n \exp[\xi^2] \left(\frac{\mathrm{d}_D}{\mathrm{d}_D\xi}\right)^n \exp[-\xi^2] \tag{37}$$

can be understood as D-deformed Hermite polynomials. In fact the polynomials $H_n^D(\xi)$ defined above are a particular case of the generalized Hermite polynomials studied in [23].

It is worth remarking that if $D = 1$ the results obtained in the present section reduce to the well-known results corresponding to the *undeformed* one-dimensional case.

5. Confined states in a *D*-deformed quantum well

In the present section we will study the motion of a particle confined in a fractional-dimensional quantum well defined by the following potential

$$V(\xi) = \begin{cases} 0 & \text{if } |\xi| < 1/2 \\ \infty & \text{otherwise} \end{cases} \tag{38}$$

where we have taken a unitary well width ($L = 1$). The corresponding Schrödinger equation may be written as

$$\left[-\frac{1}{2}\frac{\mathrm{d}_D^2}{\mathrm{d}_D\xi^2} + V(\xi)\right]\Psi_n(\xi) = E_n\Psi_n(\xi). \tag{39}$$

In terms of the introduced D-deformed calculus, the equation above can be immediately solved. The wavefunctions are given by

$$\Psi_n(\xi) = \begin{cases} A_n^{\mathrm{even}} \mathrm{COS}_D k_n\xi & \text{for even states} \\ A_n^{\mathrm{odd}} \mathrm{SIN}_D k_n\xi & \text{for odd states} \end{cases} \tag{40}$$

where

$$k_n = \sqrt{2E_n} \tag{41}$$

and $\mathrm{COS}_D\, X$, $\mathrm{SIN}_D\, X$ represent the D-deformed *cosine* and *sine* functions, respectively (see appendix). The constants A_n^{even} and A_n^{odd} are normalization factors corresponding to even and odd states respectively.

Now the eigenenergies can be easily computed from the boundary conditions

$$\mathrm{COS}_D \frac{k_n}{2} = 0 \tag{42}$$

for even states and

$$\mathrm{SIN}_D \frac{k_n}{2} = 0 \tag{43}$$

for odd states.

One should note that the D-deformed calculus allows us to express the fractional-dimensional problems in a very simple way and quite analogous to the corresponding undeformed ($D = 1$) problems.

The D-dependence of the energies corresponding to the ground state ($n = 1$) and to the first excited state ($n = 2$) is shown in figure 1. The eigenenergies increase as the dimension increases. This behaviour has also been observed in other fractional-dimensional systems[2] (see for instance [4, 14]).

[2] The energies of the fractional-dimensional harmonic oscillator and the hydrogenic atom are given by $E_n = n + D/2$ and $E_n = \frac{-4}{(2n+D-3)^2}$, respectively. In both cases the energy increases when the dimension increases.

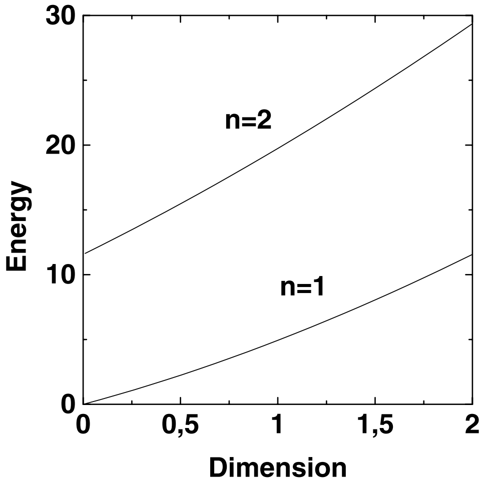

Figure 1. The eigenenergies corresponding to the ground ($n = 1$) and to the first excited ($n = 2$) states of a particle confined in a D-deformed quantum well as a function of the dimensionality.

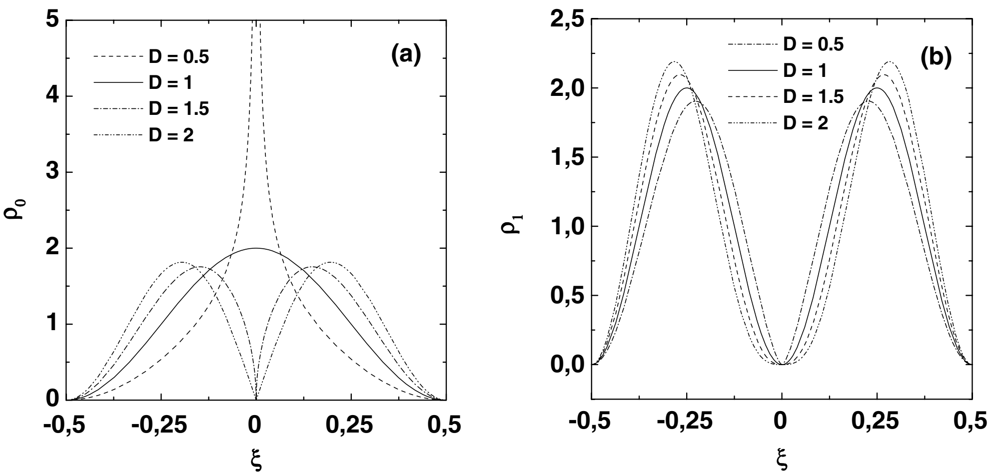

Figure 2. Position dependence of the probability density ρ_n corresponding to a particle confined in a D-deformed quantum well and for different values of the dimensional parameter: (a) for the ground state ($n = 1$) and (b) for the first excited state ($n = 2$).

In figure 2 we present the probability density

$$\rho_n(\xi) = \frac{\sigma(D)}{2} |\xi|^{D-1} |\Psi_n|^2 \tag{44}$$

corresponding to $n = 1$ (a) and to $n = 2$ (b) as a function of the pseudo-coordinate ξ for different values of the dimensionality. In both cases one can appreciate the existence of compression (spreading) of the probability density when $D < 1$ ($D > 1$). We remark that this behaviour is not a consequence of the form of the wavefunctions but of the presence of the integration weight in the probability density. Indeed, the integration weight acts as an attractive (repulsive) barrier when $D < 1$ ($D > 1$). It diverges at the origin of pseudo-coordinates when $D < 1$ (i.e. when $D < 1$ all the volume of the space is almost concentrated around $\xi = 0$) and causes a strong localization of the probability density (see figure 2(*a*)). In the case of the first excited state, however, because of the odd parity the probability density becomes zero at the origin of pseudo-coordinate and there is no longer localization in the central region.

It is worth noting that in the present case when $D > 1$, the maximum of the probability density increases as the dimension increases (see figure 2), in contrast to the behaviour observed in the fractional-dimensional harmonic oscillator (see figures 2(*b*) and 3 of [14]). This is because as the dimensionality increases, the integration weight becomes more and more repulsive favouring tunnelling through the harmonic barriers. Consequently, the particle becomes more delocalized and the maximum of the probability density decreases. In the present case, however, the quantum well is considered infinitely deep and tunnelling is suppressed. The particle is then *compressed* between the well barriers and the repulsive integration weight leading to an increase in the maximum of the probability density.

6. *D*-deformed coherent states

We now observe the spectrum problem corresponding to a fractional-dimensional annihilation operator a_D by using the rules of the D-deformed calculus. The eigenstates of a_D,

$$a_D|\alpha\rangle = \alpha|\alpha\rangle \tag{45}$$

are a D-deformation of the usual coherent states. The solution of equation (45) is given by

$$|\alpha\rangle = A_\alpha \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{[n]_D!}}|n\rangle. \tag{46}$$

From the normalization condition $\langle\alpha|\alpha\rangle = 1$, the normalization constant is found to be

$$A_\alpha = \frac{1}{\sqrt{E_D(|\alpha|^2)}}. \tag{47}$$

As usual, equation (46) can be written in terms of the vacuum state as follows

$$|\alpha\rangle = \frac{1}{\sqrt{E_D(|\alpha|^2)}} E_D(\alpha a_D^\dagger)|0\rangle. \tag{48}$$

Once we have found the expression of the D-deformed coherent states in the Fock representation, we can easily obtain its expression in the coordinate representation by using the relation

$$\Phi_\alpha(\xi) = \langle\xi \mid \alpha\rangle = \sum_{n=0}^{\infty}\langle\xi \mid n\rangle\langle n \mid \alpha\rangle. \tag{49}$$

Now by considering equations (36) and (46) we arrive at the following result

$$\Phi_\alpha(\xi) = \frac{\exp[-\xi^2/2]}{\sqrt{E_D(|\alpha|^2)}} \frac{1}{\pi^{D/4}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{2^n n!}} H_n^D(\xi). \tag{50}$$

From equation (46), the probability distribution of a D-deformed coherent state in the Fock representation is found to be

$$|\langle n \mid \alpha\rangle|^2 = \frac{1}{E_D(|\alpha|^2)} \frac{(|\alpha|^2)^n}{[n]_D!} \tag{51}$$

i.e. a D-deformation of the Poisson distribution.

7. Conclusions

Summing up, taking into account recent developments in the mathematical physics of the fractional-dimensional space and in analogy to the q-deformed calculus we have developed a new deformed calculus that we have called D-deformed calculus. Some simple fractional-dimensional systems, the free particle, the harmonic oscillator and the particle confined in a quantum well, are studied in the framework of the D-deformed quantum mechanics. Finally, the D-deformed coherent states are found.

Appendix

Here we will study the properties of some D-deformed functions. From the definition of the D-deformed derivative (equation (4)) it is easy to find that

$$\frac{\mathrm{d}_D \xi^n}{\mathrm{d}_D \xi} = [n]_D \xi^{n-1}. \tag{A1}$$

On the other hand, from the definition of the D-deformed integration (equation (22)), we also found

$$\int \xi^n \, \mathrm{d}_D \xi = \frac{\xi^{n+1}}{[n+1]_D} + \text{const.} \tag{A2}$$

In this way, one can introduce the D-deformed exponential function as follows

$$E_D(\xi) = \sum_{n=0}^{\infty} \frac{\xi^n}{[n]_D!}. \tag{A3}$$

From equation (A3) and making use of the equations (A1) and (A2) one can demonstrate in a straightforward manner that

$$\frac{\mathrm{d}_D E_D(\lambda\xi)}{\mathrm{d}_D \xi} = \lambda E_D(\lambda\xi) \qquad \lambda = \text{const.} \tag{A4}$$

and consequently

$$\int E_D(\lambda\xi) \, \mathrm{d}_D \xi = \frac{E_D(\lambda\xi)}{\lambda} + \text{const.} \tag{A5}$$

Actually, the D-deformed exponential function is a particular case of the generalized exponential function defined in [23] and can be represented as follows

$$E_D(\xi) = \exp[\xi]\Phi\left(\frac{D-1}{2}, D, -2\xi\right) \tag{A6}$$

or

$$E_D(\xi) = \Gamma(D/2)\left(\frac{\xi}{2}\right)^{1-D/2} [I_{D/2-1}(\xi) + I_{D/2}(\xi)] \tag{A7}$$

where $\Phi(a, b, x)$ and $I_\nu(x)$ are the confluent hypergeometric function and the modified Bessel function respectively.

We can also introduce D-deformed *cosine* and *sine* functions through the following definitions

$$\mathrm{COS}_D \xi = \sum_{n=0}^{\infty} \frac{(-1)^n \xi^{2n}}{[2n]_D!} = \Gamma(D/2)\left(\frac{\xi}{2}\right)^{1-D/2} J_{D/2-1}(\xi) \tag{A8}$$

and

$$\mathrm{SIN}_D\xi = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\xi^{2n-1}}{[2n-1]_D!} = \Gamma(D/2)\left(\frac{\xi}{2}\right)^{1-D/2} J_{D/2}(\xi) \tag{A9}$$

where $J_\nu(x)$ represents the Bessel function. Thus, the following identities can be easily verified

$$E_D(\pm\mathrm{i}\xi) = \mathrm{COS}_D\xi \pm \mathrm{i}\mathrm{SIN}_D\xi \tag{A10}$$

and

$$\frac{\mathrm{d}_D\mathrm{COS}_D\xi}{\mathrm{d}_D\xi} = -\mathrm{SIN}_D\xi \qquad \frac{\mathrm{d}_D\mathrm{SIN}_D\xi}{\mathrm{d}_D\xi} = \mathrm{COS}_D\xi \tag{A11}$$

It is straightforward to check that all the definitions and equations given in this appendix recover the corresponding undeformed expressions when $D = 1$, as they must.

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